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# High order stabilized finite element method for MHD plasma modeling

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- 2 Stabilization for Full MHD
- 3 Applications: Internal kink, Bohm B.C.
- 4 Conclusions and perspectives

## Motivations

- Go from Reduced to the full MHD model
- Ensure stability of flows dominated by convection
- Boundary conditions

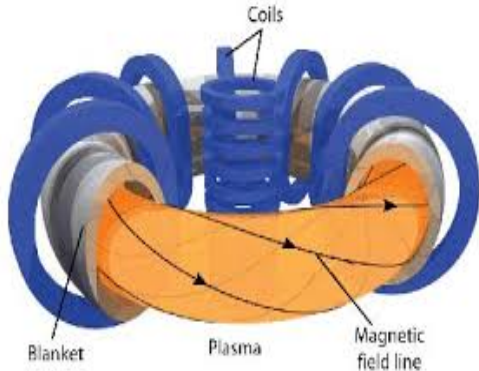
**Main goal:** use JOREK to simulate Edge-localized modes (ELMS) with the Full-MHD model

# Tokamaks

- Tokamak stands for: Toroidal chamber with magnetic coils (From Russian: Toroidal'naya Kamera s Magnitnymi Katushhami).
- Principal tokamaks at present: JET (UK), Asdex-Upgrade (Germany), Tore-Supra (France) and DIII-D (USA)
- Drifts → Nested magnetic field lines → Confinement is improved
- The amplification factor of a fusion reactor is defined as

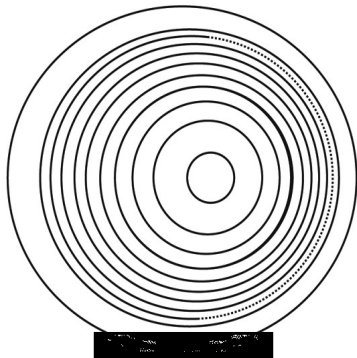
$$Q = \frac{P_{fus}}{P_{inj}}$$

- JET:  $Q \approx 1$  - ITER:  $Q = 10$
- Economically exploitable  $Q = 50$



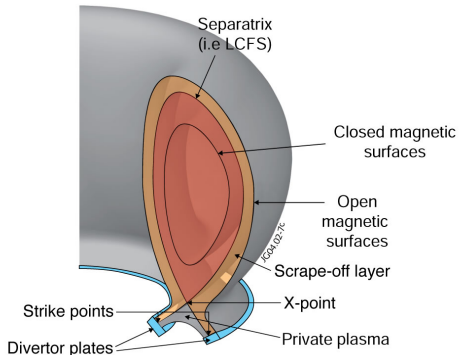
# Tokamak configurations

There are two main configurations for tokamaks



## Limiter

Material structure into the plasma



## Divertor

*X-point configuration*

Material structure placed outside

- Nonlinear reduced MHD and full MHD under implementation
- Cubic C1 finite elements (quadrangles)/Fourier (modal)
- Galerkin formulation
- Linearized Crank-Nicholson and Gear: Implicit scheme
- GMRES solver, Hybrid paralelization (MPI/OpenMPI)

# Jorek: Numerical method

- Spatial discretization: 4<sup>th</sup>-order accurate Galerkin method based on Bézier Formalism
- An arbitrary quantity is expanded on a unit square as

$$P(s, t) = \sum_{i=1}^4 \sum_{j=1}^4 p_{ij} \sigma_{ij} b_{ij}(s, t)$$

- $(s, t) \in [0, 1]$ ,  $\sigma$  assure continuity of  $P$
- Basis functions  $b_{ij}$ : Bernstein polynomials

$$b_{11} = (1-s)^2(1-t)^2(1+2s)(1+2t),$$

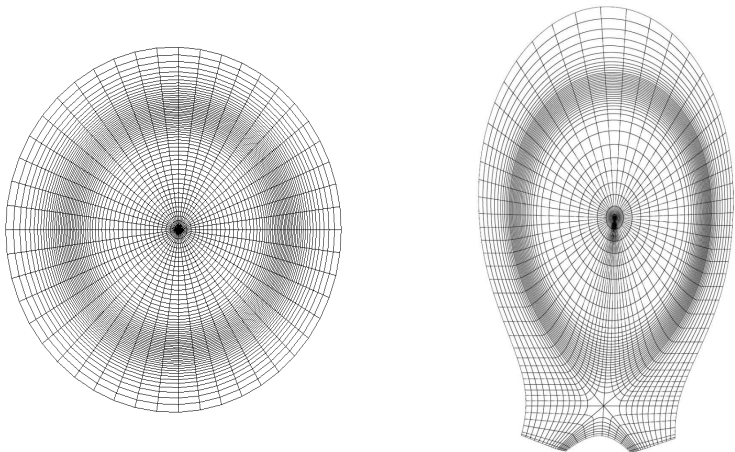
$$b_{12} = 3(1-s)^2(1-t)^2(1+2t)s,$$

$$b_{13} = 3(1-s)^2(1-t)^2(1+2s)t,$$

$$b_{14} = 9(1-s)^2(1-t)^2 st.$$



## Examples of meshes: poloidal and X-point



# System of equations: Full MHD

The single-fluid full MHD system in consideration is

$$\left\{ \begin{array}{l} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{m} = \nabla \cdot (\underline{\mathcal{D}} \nabla \rho'), \\ \frac{\partial \mathbf{m}}{\partial t} + \nabla \cdot \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} - \mathbf{B} \otimes \mathbf{B} \right) + \nabla \left( p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) = \nabla \cdot \underline{\tau}, \\ \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left( \left( \mathcal{E} + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) \frac{\mathbf{m}}{\rho} - \frac{\mathbf{B} \cdot \mathbf{m}}{\rho} \mathbf{B} \right) = \nabla \cdot \mathbf{q} + \nabla \cdot \left( \underline{\tau} \frac{\mathbf{m}}{\rho} \right), \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left( \frac{\mathbf{m} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{m}}{\rho} \right) = -\nabla \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \end{array} \right.$$

with  $\mathbf{m} \equiv \rho \mathbf{v}$ ,  $\mathcal{E} = \rho \varepsilon + \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m} + \frac{\mathbf{B} \cdot \mathbf{B}}{2}$ ,  $p = (\gamma - 1)\rho \varepsilon = \rho T$ ,

$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \underline{\eta} \mathbf{J}$ ,  $\mathbf{J} \equiv \nabla \times \mathbf{B}$ .

The ideal MHD system driving the previous system can be written in the following compact form

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot \underline{\mathbf{f}} = 0,$$

where

$$\mathbf{w} = \begin{pmatrix} \rho \\ \mathbf{m} \\ \mathcal{E} \\ \mathbf{B} \end{pmatrix}, \quad \underline{\mathbf{f}} = \begin{pmatrix} \mathbf{m}^T \\ \mathbf{v} \otimes \mathbf{m} + P\mathbb{I} - \mathbf{B} \otimes \mathbf{B} \\ \mathcal{H}\mathbf{m}^T - (\mathbf{B} \cdot \mathbf{v})\mathbf{B}^T \\ \mathbf{v} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{v} \end{pmatrix}$$

$$P = p + \frac{\mathbf{B} \cdot \mathbf{B}}{2}, \quad \mathcal{H} = \frac{\mathcal{E} + P}{\rho}$$

Quasi-linear form

$$\frac{\partial \mathbf{w}}{\partial t} + \underline{\mathbf{L}}(\mathbf{w}, \partial) \mathbf{w} = 0$$

The operator  $\underline{\mathbf{L}}(\mathbf{w}, \partial)$

$$\underline{\mathbf{L}}(\mathbf{w}, \partial) = \begin{pmatrix} 0 & \partial^T & 0 & \mathbf{0}^T \\ -\frac{\mathbf{m}}{\rho^2} \mathbf{m} \cdot \partial + \frac{\partial P}{\partial \rho} \partial & \frac{1}{\rho} (\mathbf{m} \otimes \partial + \mathbf{m} \cdot \partial) + \left( \frac{\partial P}{\partial \mathbf{m}} \otimes \mathbf{m} \right)^T & \frac{\partial P}{\partial \varepsilon} \partial & (\mathbf{B} \otimes \partial)^T - \mathbf{B} \cdot \partial + \left( \frac{\partial P}{\partial \mathbf{B}} \otimes \mathbf{m} \right)^T \\ \frac{\partial \mathcal{H}}{\partial \rho} \mathbf{m} \cdot \partial + \frac{\mathbf{B} \cdot \mathbf{m}}{\rho^2} \mathbf{B} \cdot \partial & \left( \frac{\partial \mathcal{H}}{\partial \mathbf{m}} \right)^T \mathbf{m} \cdot \partial + \mathcal{H} \partial^T - \frac{\mathbf{B}^T}{\rho} \mathbf{B} \cdot \partial & \frac{\partial \mathcal{H}}{\partial \varepsilon} \mathbf{m} \cdot \partial & \left( \frac{\partial \mathcal{H}}{\partial \mathbf{B}} \right)^T \mathbf{m} \cdot \partial - \frac{\mathbf{B} \cdot \mathbf{m}}{\rho} \partial^T - \frac{\mathbf{m}^T \mathbf{B} \cdot \partial}{\rho} \\ -\frac{\mathbf{B} \cdot \mathbf{m} \cdot \partial - \mathbf{m} \mathbf{B} \cdot \partial}{\rho^2} & \frac{\mathbf{B} \otimes \partial - \mathbf{B} \cdot \partial}{\rho} & \mathbf{0} & \frac{\mathbf{m} \cdot \partial - \mathbf{m} \partial^T}{\rho} \end{pmatrix}$$

**Seven eigenvalues:** 2 slow and 2 fast acoustic waves, 2 Alfvén waves and 1 material wave

# Weak formulation

Rewriting the system as  $\mathbf{R}(\mathbf{w}) = 0$ , in which

$$\mathbf{R}(\mathbf{w}) := \frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot \underline{\mathbf{f}} - \nabla \cdot \underline{\mathbf{g}}, \quad \underline{\mathbf{g}} : \text{Transport process}$$

## Weak formulation

$$\int_{\Omega_{x,h}} \mathbf{R}(\mathbf{w}) \cdot \mathbf{w}^* = 0, \quad \forall \mathbf{w}^* \in \vec{\mathcal{W}}_h(\Omega_{x,h})$$

## Divergence free constraint

Vector potential formulation

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{Coulomb gauge : } \nabla \cdot \mathbf{A} = 0$$

$(\nabla \cdot \mathbf{A})(\nabla \cdot \mathbf{A}^*)$  added as a penalization term

It comes to add a contribution

$$\int_{\Omega_{\mathbf{x},h}} \mathbf{R}(\mathbf{w}) \cdot \mathbf{w}^* + \int_{\Omega_{\mathbf{x},h}} \mathbf{w}' \cdot (\underline{\mathbf{L}}^T(\mathbf{w}, \partial)) \mathbf{w}^* = 0, \quad \forall \mathbf{w}^* \in \overline{\mathcal{W}}_h(\Omega_{\mathbf{x},h})$$

where  $\mathbf{w}'$  is vector of subscales and  $\underline{\mathbf{L}}^T(\mathbf{w}, \partial)$  is the adjoint of  $\underline{\mathbf{L}}(\mathbf{w}, \partial)$   
 $\mathbf{w}'$  can be approximated by

$$\mathbf{w}' \approx \mathcal{T}(\underline{\mathbf{L}}(\mathbf{w}, \partial) \delta \mathbf{w})$$

Several options for  $\mathcal{T}$ , choice based on heuristic arguments

$$\mathcal{T} = \alpha \frac{h_e}{2\|\mathbf{a}\|} \Rightarrow \frac{\|\frac{\partial \mathbf{x}}{\partial \boldsymbol{\zeta}}\|_2}{\|\boldsymbol{\Lambda}\|_\infty} \mathbb{I}$$

# Stabilized weak formulation

## Set of interpolated variables

$$\mathbf{w} = \begin{pmatrix} \rho \\ \mathbf{m} \\ \mathcal{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \frac{\rho T}{\gamma-1} + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \mathbf{B} \cdot \mathbf{B} \end{pmatrix} \Rightarrow \mathbf{Y} = \begin{pmatrix} \rho \\ \mathbf{v} \\ T \\ \mathbf{A} \end{pmatrix}$$

## Stabilized weak formulation

$$\begin{aligned} & \int_{\Omega_{\mathbf{x},h}} (\underline{\mathbf{M}}(\mathbf{Y}) \partial_t \mathbf{Y}) \cdot \tilde{\mathbf{w}}^* - \int_{\Omega_{\mathbf{x},h}} \underline{\mathbf{F}} : (\partial \odot \mathbf{w}^*) + \int_{\Omega_{\mathbf{x},h}} \underline{\mathbf{D}} : (\partial \odot \mathbf{w}^*) - \frac{1}{\epsilon} \int_{\Omega_{\mathbf{x},h}} (\nabla \cdot \mathbf{A})(\nabla \cdot \mathbf{A}^*) \\ & + \int_{\partial \Omega_{\mathbf{x},h}} \underline{\mathbf{F}} : (\mathbf{n} \odot \mathbf{w}^*) - \int_{\partial \Omega_{\mathbf{x},h}} \underline{\mathbf{D}} : (\mathbf{n} \odot \mathbf{w}^*) - \frac{1}{\epsilon} \int_{\partial \Omega_{\mathbf{x},h}} (\mathbf{n} \times \partial_t \mathbf{A}) \cdot \mathbf{B}^* + \frac{1}{\epsilon} \int_{\partial \Omega_{\mathbf{x},h}} \mathbf{S}(\mathbf{w}, \mathbf{w}_b, \mathbf{w}^*) \\ & = - \int_{\Omega_{\mathbf{x},h}} (\underline{\mathbf{L}}(\mathbf{w}, \partial) \delta \mathbf{w}) \cdot \underline{\mathbf{T}}(\underline{\mathbf{L}}^T(\mathbf{w}, \partial)) \mathbf{w}^* \end{aligned}$$

# Stabilized weak form

In the previous equation

$$\underline{\mathbf{F}} = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + P \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \\ \rho \mathcal{H} \mathbf{v} - \mathbf{B} \cdot \mathbf{v} \mathbf{B} \\ -\mathbf{v} \times \mathbf{B} \end{pmatrix}, \quad \underline{\mathbf{D}} = \begin{pmatrix} \underline{\mathcal{D}} \partial \rho' \\ \underline{\tau} \\ \underline{\tau} \mathbf{v} \\ -(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \end{pmatrix}$$

and the boundary conditions

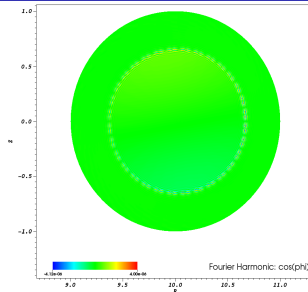
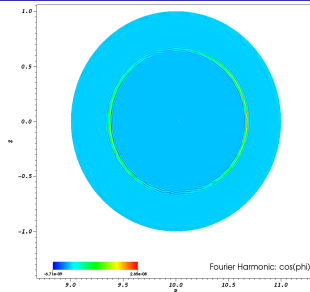
$$\mathbf{S}(\mathbf{w}, \mathbf{w}_b, \mathbf{w}^*) = \begin{pmatrix} (\rho - \rho_b) \rho^* \chi_\rho \\ \rho(\mathbf{v} - \mathbf{v}_b) \cdot \mathbf{m}^* \chi_{\mathbf{v}} + (\rho \mathbf{v} - \mathbf{m}_b) \cdot \mathbf{m}^* \chi_m \\ \rho(T - T_b) \mathcal{E}^* \chi_T + (\rho - \rho_b) \mathcal{E}^* \chi_p \\ (\mathbf{A} - \mathbf{A}_b) \cdot \mathbf{J}^* \chi_{\mathbf{A}} + (\mathbf{B} - \mathbf{B}_b) \cdot \mathbf{B}^* \chi_{\mathbf{B}} \end{pmatrix}$$

For further details, please refer to the Research Report: [hal.inria.fr/hal-01294788](http://hal.inria.fr/hal-01294788)

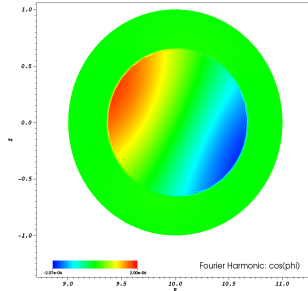
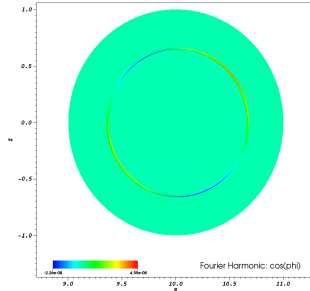


# Resistive internal kink

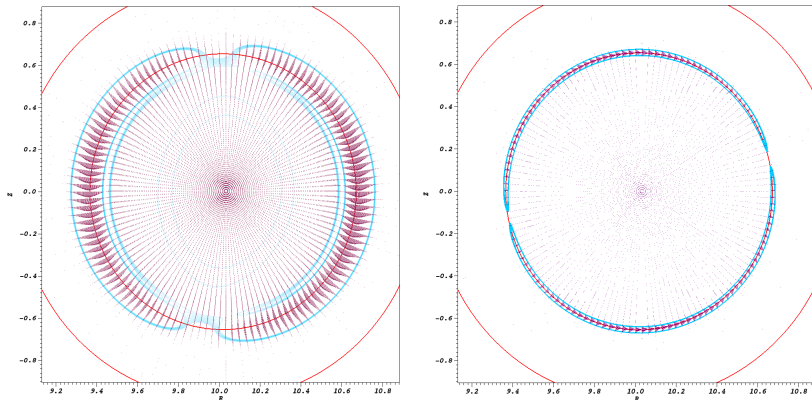
Numerical  
viscosity



VMS-  
Stabilization



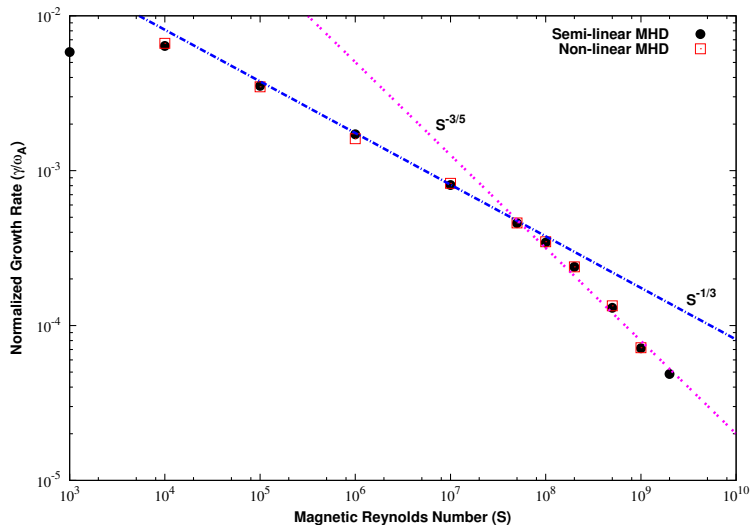
# Resistive internal kink dynamics



Structure layer:  $S = 10^5 \approx 10^{-1}$  and  $S = 10^8 \approx 2 \cdot 10^{-2}$

$S \propto \eta^{-1}$ : Magnetic Reynolds number.

# Resistive internal kink: growth rate



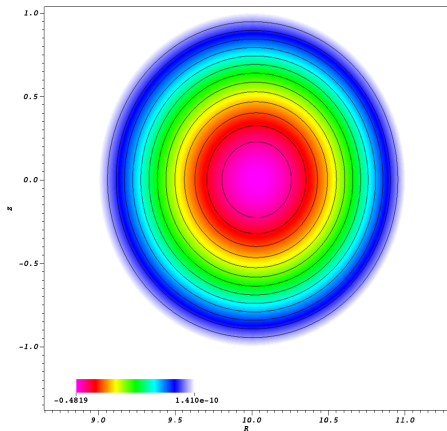
$$\gamma \propto S^{-1/3} \rightarrow \text{Kink}$$

$$\gamma \propto S^{-3/5} \rightarrow \text{Tearing}$$

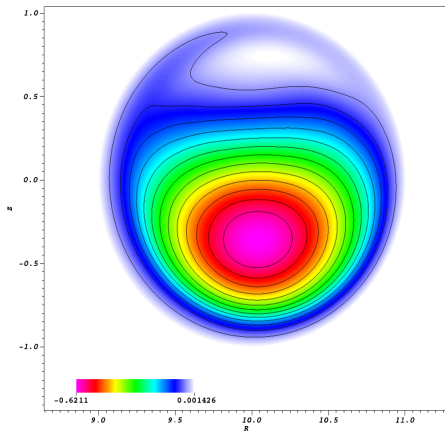
$$\text{Threshold} \approx S = 10^8$$

# Resistive internal kink: magnetic reconnection

$t = 0\tau_A$

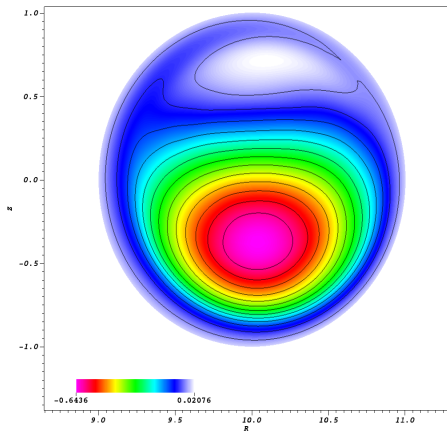


$t = 4400\tau_A$

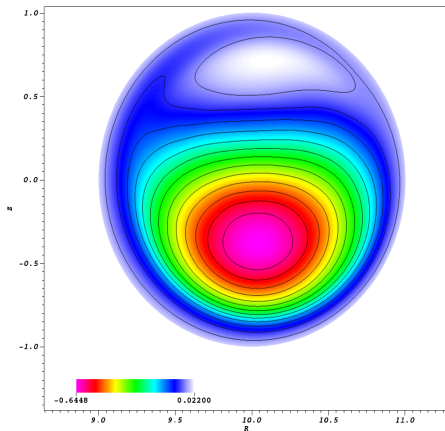


# Resistive internal kink: magnetic reconnection

$t = 4460\tau_A$

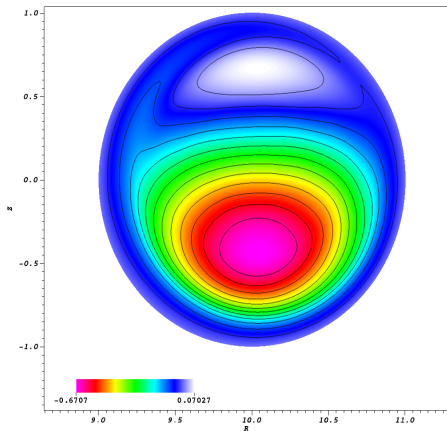


$t = 4464\tau_A$

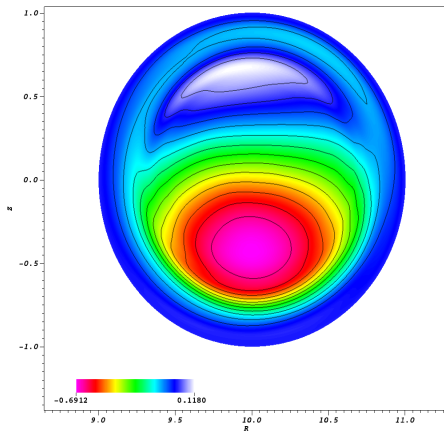


# Resistive internal kink: magnetic reconnection

$t = 4648\tau_A$



$t = 5484\tau_A$



# Boundary Conditions: Bohm

We need to compute the following integrals at the boundary

$$- \int_{\partial\Omega_{x,h}} \pm \rho \frac{\sqrt{\gamma T}}{\|\mathbf{B}\|} \mathbf{B} \cdot \mathbf{n} \rho^*,$$

$$- \int_{\partial\Omega_{x,h}} \left( \pm \rho \mathbf{v} \frac{\sqrt{\gamma T}}{\|\mathbf{B}\|} \mathbf{B} \cdot \mathbf{n} \right) \cdot \mathbf{m}^*,$$

$$- \int_{\partial\Omega_{x,h}} \pm (p\mathbf{n} + \pi\mathbf{n} - \mathbf{B}(\mathbf{B} \cdot \mathbf{n})) \cdot \mathbf{m}^*$$

$$- \int_{\partial\Omega_{x,h}} \pm p \frac{\sqrt{\gamma T}}{\|\mathbf{B}\|} \mathbf{B} \cdot \mathbf{n} \mathcal{E}^*$$

Where  $\mathbf{n}$  is the outward normal  
and

$$\pm \equiv \frac{\mathbf{B} \cdot \mathbf{n}}{\|\mathbf{B} \cdot \mathbf{n}\|}$$

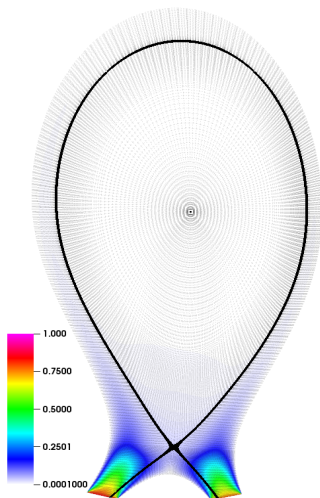
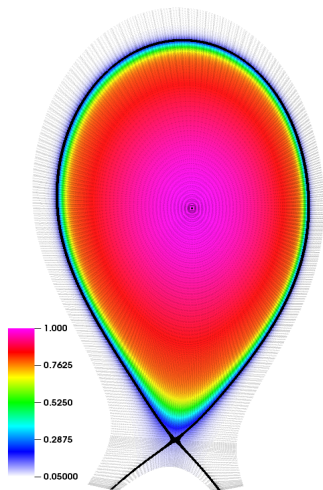
so that  $\mathbf{v}_b \cdot \mathbf{n}$  is always positive

Also, a penalization term

$$\frac{1}{\epsilon} \int_{\partial\Omega_{x,h}} \left( \rho \mathbf{v} \mp \rho \sqrt{\gamma T} \frac{\mathbf{B}}{\|\mathbf{B}\|} \right) \cdot \mathbf{m}^*$$

# X-point results: Initial density and velocity

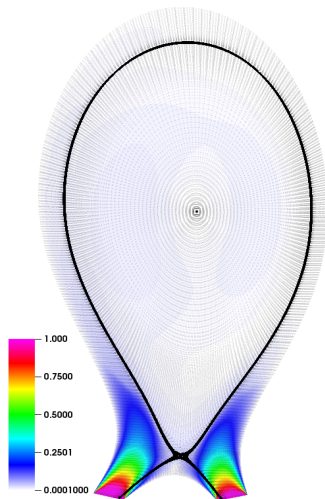
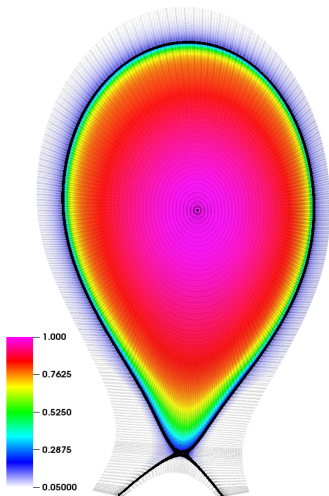
$$t = 1\tau_A$$





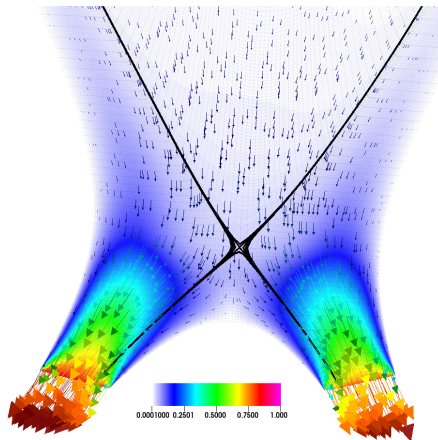
# X-point results: Initial density and velocity

$$t = 400\tau_A$$

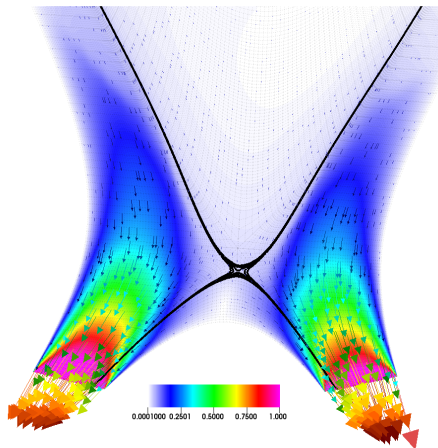


# X-point results: Velocity field

$t = 1\tau_A$

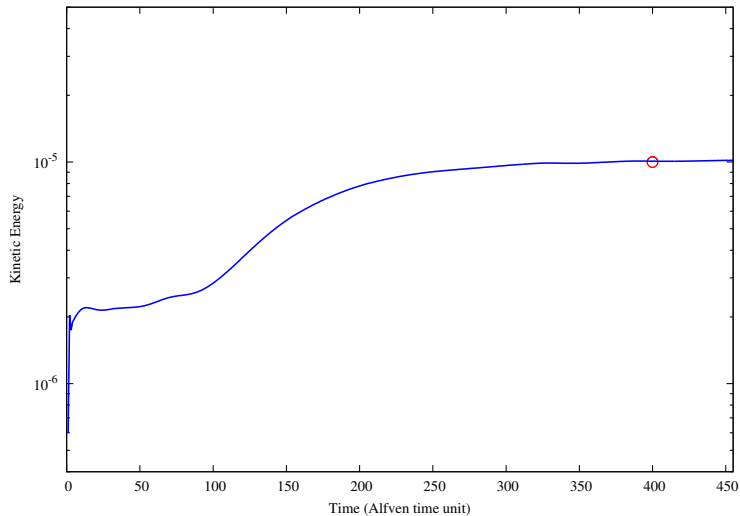


$t = 400\tau_A$



# X-point results: Quasi-steady state

Evolution of the Kinetic Energy:  $n = 0$



## Conclusions

- Full MHD equations successfully implemented for simulating **circular** plasmas
- Stabilization scheme has improved the quality of the results
- Quasi-steady state for X-point geometry with Bohm B.C.

## Perspectives

- Simulation of ELMs using the full model